Asset-Liability Management in Private Wealth Management

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Over the past decade, private wealth management has become a profitable business for banks and asset managers around the globe. According to the private banking and wealth management survey conducted by *Euromoney* in 2008, global private banking assets rose to US$ 7.6 trillion in 2008 from US$ 3.3 trillion the year before. The increase in assets currently driving the wealth management market is creating greater opportunities for wealth advisors to leverage new technology in order to acquire new clients and grow profits. As a result, competition is increasing among wealth advisory firms to improve existing client relationships and provide new tools to improve advisor effectiveness. While the private banking industry is relatively well equipped on the tax-planning side with tools to analyze the situation of high-net-worth individuals operating offshore or across multiple tax jurisdictions, the software packages used on the financial-simulation side typically suffer from significant limitations and cannot satisfy the needs of a sophisticated clientele.

Most existing financial software packages used by private bankers to generate asset allocation recommendations rely on single-period mean-variance portfolio optimization, which cannot yield a proper strategic allocation for at least two reasons. On the one hand, optimization parameters (expected returns, volatilities, and correlations) are defined as constant across time, a practice that is contradicted by empirical observation and does not allow for the length of the investment horizon to be taken into account. On the other hand, and perhaps more importantly, liability constraints and the risk factors affecting them, such as inflation risk on targeted spending, are neither modeled nor explicitly taken into account in the portfolio construction process. Overall, the process involved in dealing with a private client typically leads to a detailed analysis of the client's objectives, constraints, and risk aversion parameters, sometimes on the basis of rather sophisticated approaches. Yet, it is striking that once this information has been collected, and sometimes formalized, little is done in terms of customizing a portfolio solution to the specific needs of the client. Typically, the approach consists of providing several profiles expressed in terms of volatility or drawdown levels, with a specification of how the capital will eventually be accessed (e.g., annuity or lump-sum payment). The client's specific objectives, constraints, and associated risk factors however, are not taken into account in the design of the optimal allocation. While some industry players have recently developed planning tools that model assets in a multiperiod stochastic framework, asset-liability matching for individuals remains an area for exploration. The objective of this article is to shed some light on new forms of welfare-improving financial innovation inspired by asset-liability management techniques that were originally introduced in the context of institutional money management, but now have applications within a private wealth management context.

Asset-liability management (ALM) denotes the adaptation of the portfolio management process to handle the presence of various constraints relating to the liabilities of an investor. We argue that suitable extensions of portfolio optimization techniques used by institutional investors (e.g., pension funds) would be useful in the context of private wealth management, because they have precisely been engineered to incorporate an investor's specific constraints, objectives, and horizon into the portfolio construction process. The investor's constraints can be summarized in terms of a single state variable, the value of the *liability* portfolio. Although ours is a fairly stylized model and important effects, such as taxes or mortality risk, are not explicitly taken into account at this stage (see our suggestions for further research), we believe it represents a significant normative step toward a better understanding of private wealth management decisions. In this context, our article can be regarded as an attempt to provide a first step toward a rational framework for private investors' financial decisions, which extends standard portfolio optimization techniques by recognizing that the various aforementioned factors seriously affect the optimal allocation decision.

Our main contribution is to show that a significant fraction of the complexity of optimal asset allocation decisions for private investors can conveniently be captured through the introduction of a single state variable—the liability value—that can account in a parsimonious way for investors' specific constraints and objectives. Within the framework of private wealth management, we use a broad definition of liabilities that encompasses any commitment or spending objective, typically self-imposed as opposed to exogenously imposed as in a pension fund context, that an investor is facing. An example for an investor committed to a real estate acquisition would be a future commitment, or soft liability, for which money must be available when needed.

Overall, it is not the performance of a particular fund nor that of a given asset class that will be the determining factor in the ability to meet a private investor's expectations. The success or failure of meeting the investor's long-term objectives is dependent on an ALM exercise that aims to determine the proper strategic inter-class allocation as a function of the investor's specific objectives, constraints, and time horizon. In other words, the decisive factor will be the ability to design an asset allocation solution that is a function of the particular risks to which the investor is exposed, as opposed to the market as a whole. Similarly, the concept of the risk-free asset is a function of household time horizon and objective. Hence, a five-year zero-coupon Treasury bond will not be a perfectly safe investment for a private investor interested in a real estate acquisition five years in the future. The actual risk-free asset in this context (the *liability-hedging portfolio*) would instead be an asset perfectly correlated with real estate prices. More generally, an investor whose objective is related to the acquisition of property would accept low and even negative returns in situations when real estate prices significantly decrease, but would not be satisfied with relatively high returns if such returns were not sufficient to meet a dramatic increase in real estate prices. In such circumstances, a long-term investment in stocks and bonds, which are weakly correlated with real estate, would not be the right investment solution. In a pension context, absolute return performance, often perceived as a natural choice in the context of private wealth management, would not be satisfactory for a private investor facing long-term inflation risk, where the concern is capital preservation in real, as opposed to nominal, terms. In other words, the first benefit of the ALM approach is its impact on the menu of asset classes, with a focus on assets that exhibit the highest possible correlation to the liability portfolio.

Our article is related to the literature on long-term financial decisions, beginning with the seminal work of Merton [1969, 1971] that was further specialized to encompass either uncertain interest rates (Campbell and Viceira [2001], Brennan and Xia [2002], and Wachter [2003]), or uncertain risk premia (Kim and Omberg [1996] and Campbell and Viceira [1999]), or both (Brennan, Schwartz, and Lagnado [1997], Lynch [2001], and Campbell, Chan, and Viceira [2003]). These early articles highlight some important aspects of life-cycle investing, including the usefulness of real bonds for inflation-hedging purposes, but, they ignore some of the key complexities of private financial decisions. As emphasized in his AFA presidential address, Campbell [2006] stated that "the study of household finance is challenging because households face a number of constraints not captured by textbook models."

A large number of more recent articles have subsequently focused on integrating various salient features of private wealth management, including the impact of human capital (see, for example, Bodie, Merton, and

Samuelson [1992], Heaton and Lucas [2000], Viceira [2001], and Cocco, Gomes, and Maenhout [2005]), illiquid real estate allocation (Sinai and Souleles [2005]), and borrowing constraints on optimal allocation decisions. However, because they typically rely on standard expected utility maximization of terminal wealth, these articles fail to integrate a key dimension of private wealth management, namely, that investment decisions should be designed to help investors achieve certain predetermined objectives, such as preparing for retirement or, earlier in the life cycle, preparing for real estate acquisition. In a nutshell, we argue that existing literature on household finance has mostly taken an asset management perspective as opposed to an asset-liability management perspective.[1] Several authors have attempted to extend intertemporal selection analysis to account for the presence of liability constraints in asset allocation policy. A first attempt toward the introduction of liability constraints in optimal portfolio selection theory was made by Merton [1993], who studied the allocation decision of a university that manages an endowment fund. Two examples of this particular strand of the finance literature are Rudolf and Ziemba [2004], who formulated a continuous-time dynamic programming model of pension fund management in the presence of a time-varying opportunity set, and Sundaresan and Zapatero [1997], whose work also involved an endogenous retirement decision.

Our work is an attempt to merge these two somewhat separate strands of the literature, namely, the literature on long-term financial decisions for private investors, which has primarily adopted an asset-only perspective, and the literature on asset-liability management decisions, which has primarily been analyzed from an institutional perspective (e.g., pension funds, insurance companies, or endowments). We cast the long-horizon life-cycle investment problem within an asset-liability management framework suitable for private wealth management, in order to show that pursuing an asset-only strategy generally involves a substantial opportunity cost. Broadly speaking, adopting an ALM approach leads to defining risk and return in relative terms with respect to the liability portfolio, which is a critical improvement over asset-only allocation models that fail to account for the presence of various investment and/or consumption goals and objectives, such as preparing for retirement or for a real estate acquisition. As a result, adopting an ALM approach leads to a focus on the liability-hedging properties of various asset classes, a focus that is, by definition, absent from an asset-only perspective.

The article is organized as follows. In the next section, we introduce a formal stylized model of asset-liability management for household financial decisions. Then, we present illustrations of the usefulness of asset-liability management techniques within a household finance context, focusing on a pension objective and a real estate acquisition objective. The conclusion, as well as suggestions for further research, are in the last section.

A FORMAL MODEL OF ASSET-LIABILITY MANAGEMENT IN PRIVATE WEALTH MANAGEMENT

In this section, we will introduce a formal model of asset-liability management and discuss its application within a private wealth management context. This analytical approach to ALM is appealing in spite of its highly stylized nature because it leads to a tractable solution, allowing a full and explicit understanding of the various mechanisms affecting the optimal allocation strategy. In particular, the three-fund separation theorem we obtain, typical of optimal asset allocation decisions in the presence of stochastic state variables, is a parsimonious way to capture some of the complexity involved in private wealth management decisions.

Stochastic Model for Risk Factors Impacting Asset and Liability Values

Based on a suitable econometric model, we provide estimates for the short- and long-term dependencies between the return on a set of asset classes and factors impacting liability values. We rely on a stationary vector-autoregressive (VAR) approach for modeling the joint asset and liability return dynamic distributions, similar to Campbell, Chan, and Viceira [2003] and Campbell and Viceira [2005], extended to an asset-liability management (see Hoevenaars et al. [2008]) framework,

$$z_{t+1} = \Phi_0 + \Phi_1 z_t + \varepsilon_t \tag{1}$$

with z_t being a vector of risk factors impacting asset and liability values, and ε_t an error term whose distribution needs to be specified.

On the asset side, our empirical analysis focuses on a set of traditional and alternative asset classes. Stock returns are represented by the CRSP Value-Weighted Index. Commodities are proxied by the S&P Goldman Sachs Commodity Index (GSCI). Real estate investments are represented by the FTSE NAREIT U.S. Real Estate Index, which is a value-weighted basket of REITs listed on the NYSE, AMEX, and NASDAQ. We thus limit the opportunity set to liquid and publicly traded assets. Finally, we add the returns of the 20-year constant-maturity bond and three-month U.S. Treasury bill, both from the CRSP database. Following the evidence from the extensive literature on return predictability (see Stock and Watson [1999], among others), we also add potential predictive economic variables to the set of endogenous variables by introducing the dividend yield (see, for example, Campbell and Shiller [1988], Hodrick [1992], and Campbell and Viceira [2002]), the credit spread (the difference between Moody's Seasoned Baa Corporate Bond Yield and the 10-year Treasury Constant Maturity Rate), and the term spread (the difference between the 10-year Treasury Constant Maturity Rate and the three-month T-Bill rate). The dividend series for the value-weighted stock index is obtained from CRSP and the dividend yield is reconstituted as the cumulated dividends received over the preceding four quarters divided by the current price. All other economic variables are from Datastream.

On the liability side, we first consider a pension objective with a focus of maintaining a target level of purchasing power after retirement. As far as this objective is concerned, the natural proxy for liability returns is the return on a Treasury inflation-protected security (TIPS), as the payoff of TIPS is governed by the cumulative inflation over the time horizon. Unfortunately, an empirical time series with a sufficiently long history is not available for TIPS. We therefore construct a time series for a constant-maturity Treasury inflation-protected bond by using the following inputs: constant-maturity nominal bond returns (r^b) from CRSP, the median inflation forecast ($\hat{\pi}$) from the Survey Research Center at the University of Michigan, and realized inflation (π) as proxied by the U.S. Consumer Price Index. The return on the TIPS portfolio is denoted by r_t^r and given by

$$r_t^r = r_t^b - \hat{\pi}_t + \pi_t - \Lambda_\pi \tag{2}$$

where Λ_π is the inflation risk premia assumed to be 50 basis points, which is consistent with Kothari and Shanken [2004]. Note that the constructed nominal return incorporates an interest rate risk premium as the nominal bond return, r^b, accounts for the presence of a term spread. The constant time-to-maturity for the bond is five years, and the five-year-ahead inflation forecast is consistently used.[2]

The second example on the liability side relates to real estate acquisition by an investor. We assume that the liability return exactly coincides with the return on the FTSE NAREIT Index. Obviously, this is a simplification because the value of a unique property will not be perfectly correlated with the return on a broad REIT index, which contains equity market exposure in addition to real estate market exposure. We chose an investable proxy for real estate returns because it will prove useful in the empirical section to analyze a complete market setting in which the risk factors in liability returns are entirely spanned by existing securities.

The vector of endogenous variables z in Equation (1) contains 10 elements. Exhibit 1 presents summary statistics for the corresponding time series of quarterly returns from 1962 Q1 through 2005 Q4. Sharpe ratios for all asset classes are somewhat similar over the sample period with the notable exception of long bonds, which are dominated on a risk-adjusted basis. The summary statistics refer to arithmetic averages over the empirical sample, but the analysis that follows is designed to derive horizon-dependent moments for the different asset classes.

To estimate the VAR model, we follow the procedure of Hoevenaars et al. [2008], who imposed restrictions on the parameters. We begin by defining a core system of equations in the VAR: consider the subvector z_1 containing the excess returns on stocks and the long bond, the return on the T-bill, the inflation rate, the credit

EXHIBIT 1
Summary Statistics

	Average Return	Volatility	Sharpe Ratio
Stocks	12.31	17.33	0.71
20Y Bond	6.19	10.29	0.60
Commodities	14.71	19.10	0.77
Real Estate	12.10	16.95	0.71
TIPS	5.00	6.23	0.80
3M T-Bill	6.82	1.51	–

Note: Summary statistics are calculated from quarterly log returns from 1962 Q1 through 2005 Q4 for stocks, bonds, and T-Bills; from 1970 Q2 through 2005 Q4 for commodities; from 1972 Q2 through 2005 Q4 for real estate; and from 1979 Q2 through 2005 Q4 for TIPS. All returns are in excess of T-bills except the T-bill returns. Average returns and volatilities are corrected for Jensen's inequality and are in annualized percentage numbers.

spread, the term spread, and the dividend yield. We estimate an unrestricted VAR model for z_1 as

$$z_{1,t+1} = \Phi_{0,1} + \Phi_{1,1} z_{1,t} + \varepsilon_{1,t}$$

As a consequence, the variables that are not contained in the core model have no impact on core variables. We then define z_2 as the complement of z_1: z_2 thus contains the excess returns on commodities, real estate, and the TIPS. We then estimate the following regression system:

$$z_{2,t+1} = \Phi_{0,2} + A z_{1,t+1} + B z_{1,t} + \Phi_{1,2} z_{2,t} + \varepsilon_{2,t}$$

where the matrix $\Phi_{1,2}$ is restricted to be diagonal as is the covariance matrix of the error term, ε_2. These restrictions imply that non-core variables have no effect on each other and that their innovations are independent. The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are not restricted. In particular, $\boldsymbol{A}$ accounts for contemporaneous correlations between non-core variables and core variables, so that we can assume the error terms, ε_1 and ε_2, are uncorrelated. Finally, we can write the restricted parameters of the VAR system (1) as

$$\Phi_0 = \begin{pmatrix} \Phi_{0,1} \\ \Phi_{0,2} \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \Phi_{1,1} & 0 \\ B + A\Phi_{1,1} & \Phi_{1,2} \end{pmatrix},$$

$$\Sigma_\varepsilon = \begin{pmatrix} \Sigma_{\varepsilon,1} & \Sigma_{\varepsilon,1} A' \\ A\Sigma_{\varepsilon,1} & \Sigma_{\varepsilon,2} + A\Sigma_{\varepsilon,1} A' \end{pmatrix}$$

These restrictions allow us to make use of the full history of core variables, which covers the period from 1962 Q1 through 2005 Q4, even if the sample of non-core variables is much smaller. Indeed, our return series for commodities begins in 1970 Q2, while the return series for real estate begins in 1972 Q2, and the series for TIPS starts in 1979 Q2.[3]

We present the estimated matrix Φ_1 in Exhibit 2 and the residual correlations in Exhibit 3. The VAR-implied dynamics for asset and liability returns are indeed governed by these two parameter matrices. The VAR modeling framework is particularly convenient in a portfolio context, since it generates analytic expressions for time-dependent variances and expected returns. The model-implied first and second moments of compounded returns can, in fact, be written as

$$\mathbb{E}_t\left(\sum_{k=1}^{T} H z_{t+k}\right) = \begin{pmatrix} \mu_T^A \\ \mu_T^L \end{pmatrix} = \left[\sum_{k=0}^{T-1} (T-k) H \hat{\Phi}_1^k\right] \Phi_0 + \left[\sum_{k=1}^{T} H \hat{\Phi}_1^k\right] z_t \quad (3)$$

$$\mathbb{V}_t\left(\sum_{k=1}^{T} H z_{t+k}\right) = \begin{pmatrix} \Sigma_T^A & \sigma_T^{AL} \\ \sigma_T^{AL'} & \sigma_T^L \end{pmatrix} = \sum_{k=1}^{T}\left[\left(\sum_{i=1}^{k-1} H\hat{\Phi}_1^i\right)\Sigma_\varepsilon \left(\sum_{i=1}^{k-1} H\hat{\Phi}_1^i\right)'\right] \quad (4)$$

EXHIBIT 2
Estimated VAR—Φ_1

		1	2	3	4	5	6	7	8	9	10
1	Stocks	−0.06	0.17	−1.74	−0.71	1.40	0.28	0.06			
2	20Y Bond	−0.05	−0.07	0.55	0.44	−0.89	1.64	−0.02			
3	3M T-Bill	−0.00	−0.03	0.76	0.01	0.07	−0.11	0.00			
4	CPI	0.00	−0.03	0.00	0.26	0.24	−0.22	0.01			
5	Credit Spread	−0.00	−0.01	0.14	0.07	0.81	0.06	−0.00			
6	Term Spread	−0.01	−0.01	−0.11	−0.00	0.46	0.70	0.00			
7	Dividend Yield	−1.09	0.03	−0.94	0.20	−0.89	0.25	1.00			
8	Commodities	−0.01	0.01	−4.55	0.11	−1.76	−1.40	0.04	−0.03		
9	Real Estate	−0.06	−0.06	−0.14	0.57	3.53	1.60	0.01		−0.04	
10	TIPS	−0.06	−0.03	0.89	−0.39	−0.20	0.62	−0.01			−0.28

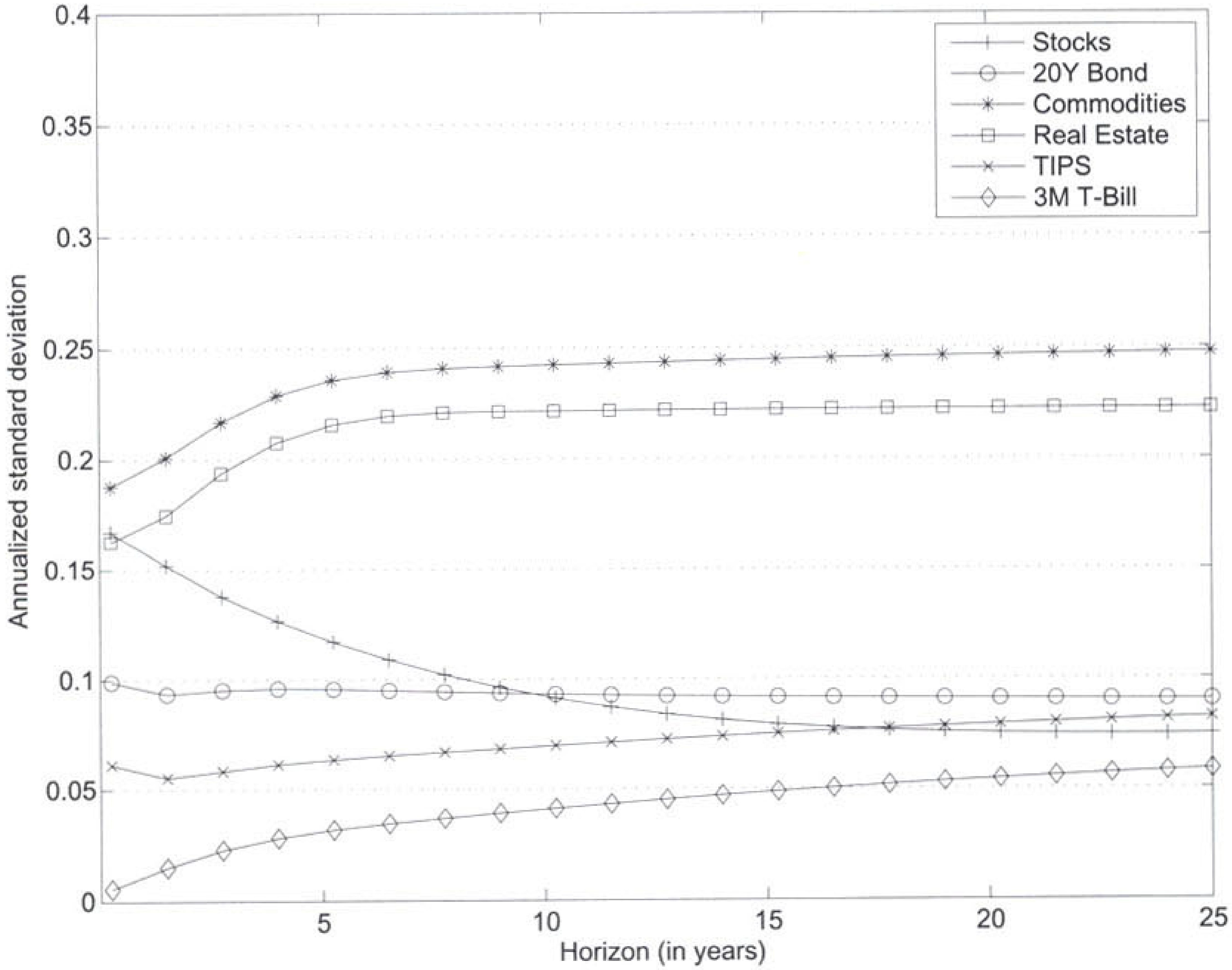
Note: Quarterly returns from 1962 Q1 to 2005 Q4 are fitted to the VAR system in Equation (1). Blank elements are zero by construction of the VAR. All returns related to tradable assets 1, 2, 8, 9, and 10 are in excess of T-bills.

EXHIBIT 3
Correlation Matrix of Residuals ε

		1	2	3	4	5	6	7	8	9	10
1	Stocks	0.08									
2	20Y Bond	0.20	0.05								
3	3M T-Bill	−0.12	0.28	0.00							
4	CPI	−0.27	−0.34	0.06	0.01						
5	Credit Spread	−0.00	0.67	0.22	−0.17	0.00					
6	Term Spread	0.00	0.07	0.23	−0.17	−0.03	0.01				
7	Dividend Yield	−0.02	−0.08	0.02	0.27	−0.13	−0.15	0.02			
8	Commodities	0.11	0.00	−0.16	−0.22	−0.06	0.06	−0.02	0.09		
9	Real Estate	0.05	0.10	0.11	0.02	0.09	0.07	0.03	−0.02	0.08	
10	TIPS	−0.18	−0.00	0.12	0.18	−0.05	0.01	−0.14	−0.06	−0.00	0.03

Note: Quarterly returns from 1962 Q1 to 2005 Q4 are fitted to the VAR system in Equation (1). Off-diagonal elements are correlations, and diagonal elements are standard deviations. All returns related to tradable assets 1, 2, 8, 9, and 10 are in excess of T-bills.

EXHIBIT 4
Term Structure of Risk



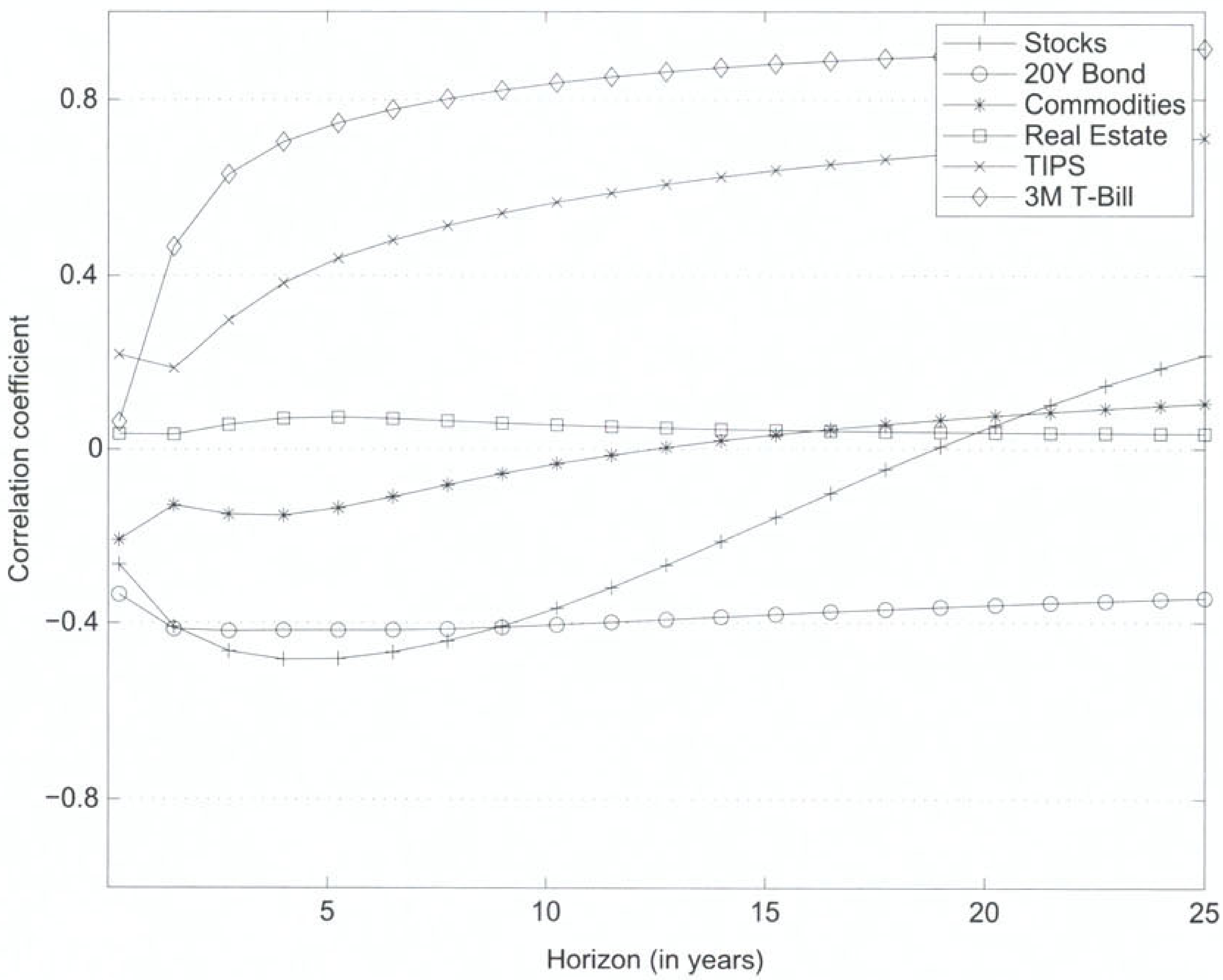
Note: This exhibit plots the volatility implied by the VAR model of the nominal return on each asset class, as a function of investment horizon.

Here, μ_T^A is the vector of expected returns on assets, and μ_T^L is the (scalar) return on the liability portfolio (inflation-linked or real estate–related) for a horizon equal to T years. Σ_T^A is the asset covariance matrix for the T-year horizon, σ_T^{AL} is the vector of covariances between the asset classes and the liability portfolio, and σ_T^L is the liability variance for a horizon equal to T years. Σ_ε denotes the residual covariance matrix. Finally, $\boldsymbol{H}$ is a matrix that selects the vector of excess returns on the assets and the liability from the state vector $\boldsymbol{z}$.

Exhibit 4 depicts the horizon-dependent annualized volatilities derived from the fitted parameters and implied by the VAR system (see Equation (4)). Consistent with the findings of Campbell and Viceira [2001], we find that equity markets are less risky in the long term. This effect is explained by the presence of implied mean-reversion in stock returns. Indeed, dividend yields are widely documented to exhibit significant predictive power for stock returns (see, for example, Campbell and Viceira [2001]). On the one hand, innovations to dividend yields and stock returns are negatively correlated (see Exhibit 3). On the other hand, lagged dividend yields are positively correlated with contemporaneous stock returns (see Exhibit 2), which leads to a smoothing effect for past innovations. In contrast, we find that investing in T-bills generates a higher annualized volatility as the investment horizon increases due to the uncertainty involved in rolling over short-term debt in the presence of stochastic interest rates. Exhibit 4 also shows that the term structure of volatilities implied by the VAR model is upward sloping for real estate and commodity investments.

Exhibit 5 displays the correlations of the various asset classes with the Consumer Price Index as functions of time horizon. It appears that the inflation-hedging properties of

Exhibit 5
Term Structure of Correlations with Realized Inflation



Note: This exhibit plots the correlation implied by the VAR model between nominal returns on the asset classes and realized inflation, as a function of investment horizon.

the various asset classes depend on the horizon. For instance, T-bills have a negative correlation with realized inflation over short horizons, but the correlation becomes positive and relatively high over horizons exceeding 10 years. Stocks display a negative correlation with inflation in the short run, but have good inflation-hedging properties over horizons that exceed 20 years. Our finding of a negative short-term relationship between expected stock returns and expected inflation is consistent with previous empirical findings on the subject (see Fama and Schwert [1977], Gultekin [1983], and Kaul [1987], among others) and is also consistent with the intuition that higher inflation leads to lower economic activity, thus depressing stock returns (e.g., Fama [1981]). In contrast, higher future inflation leads to higher dividends and thus higher returns on stocks (Campbell and Shiller [1988]), and thus equity investments should offer significant inflation protection over longer horizons, a fact that has also been confirmed by a number of recent empirical academic studies (Boudoukh and Richardson [1993] and Schotman and Schweitzer [2000]).

Overall, these findings suggest that utilizing a standard one-period optimization model, as is typically done in private wealth management, is an oversimplification that does not allow investors to benefit from life-cycle effects induced by time-varying opportunity sets. In the next subsection we will formally confirm that optimal allocation decisions are a function of time horizon and cannot be captured in the context of a static portfolio optimization exercise. We will also argue that not taking into account the presence of liabilities leads to a substantial opportunity cost.

Life-Cycle Investment Decisions in Asset-Liability Management

In a dynamic asset allocation model, it is customary to assume that the preferences of the investor are expressed in terms of expected utility of terminal asset value,

$$\max_{\omega} \mathbb{E}[u(A_T)] \tag{5}$$

where T is the investor's time horizon. One problem with this objective is that it fails to recognize that targeted (liability) payments are scheduled beyond horizon T. One approach to this problem is recognizing that terminal wealth at horizon T is a combination of a long position in the asset portfolio with value A_T and a short position in the liability portfolio with value L_T. Another approach to account for the presence of liability payments beyond the horizon is the introduction of a related state variable, the funding ratio, defined as the ratio of assets to liabilities,

$$F_t \equiv \frac{A_t}{L_t} \tag{6}$$

which is well defined if L_t is not zero. This quantity is commonly used in pension fund practice, where a fund is said to be overfunded when the funding ratio is greater than 100%, to be fully funded when the funding ratio equals 100%, and to be underfunded when the funding ratio is lower than 100%. From an interpretation standpoint, focusing on the funding ratio amounts to using the liability value process $(L_t)_{t\geq 0}$ as opposed to the bank account for the numeraire, an approach that has been used by van Binsbergen and Brandt [2007], and which we adopt in our analysis.

We take u to be the constant relative risk aversion (CRRA) utility function, defined as

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} \quad \text{for } x > 0$$
$$u(x) = -\infty \quad \text{for } x \leq 0$$

where γ lies in $[1, \infty]$. If $\gamma = 1$, we obtain the logarithmic utility function.

With the additional assumption of log-normal return distributions, the portfolio choice model collapses into a mean-variance problem,

$$\max_{\omega}\left\{\mathbb{E}\left[\frac{F_T^{1-\gamma}}{1-\gamma}\right]\right\} \Rightarrow \max_{\omega}\left\{\mathbb{E}[r_T^F] + \frac{1-\gamma}{2}\mathbb{V}[r_T^F]\right\} \tag{7}$$

with r_T^F as the T-period forward-looking log funding ratio return such that $F_T = \exp(r_T^F)$ and γ as the level of relative risk aversion. In the absence of log-normal returns, this approach can also be justified by a second-order approximation (see Campbell [1993, 1996] or Campbell, Chan, and Viceira [2003] for details).

Analyzing the preceding program for different levels of relative risk aversion coincides with analyzing the mean-variance efficient frontier of the terminal funding ratio or the T period funding ratio return. Portfolios on the efficient frontier are thus the solutions to the following program:

$$\min_{\omega} \frac{1}{2} \mathbb{V}\,(r_T^F) \\ s.t.: \mathbb{E}\,(r_T^F) = \mu_T^F \tag{8}$$

where μ_T^F is an achievable target funding ratio return. The Lagrangian for this program is given as

$$L = \frac{1}{2} \mathbb{V}\,(r_T^F) - \lambda(\mathbb{E}\,(r_T^F) - \mu_T^F) \tag{9}$$

Transparent from Equations (3) and (4), the first and second moments in Equation (9) can be derived from the VAR model and exhibit an explicit dependency with respect to time horizon. Assuming a fixed-mix allocation ω for a given time-to-maturity T, and discretizing Ito's lemma (see details in Campbell, Chan, and Viceira [2003]), we can write the log funding ratio return as

$$r_T^F = \omega' \left(r_T^A + \frac{1}{2}\sigma_T^A \right) - \frac{1}{2}\omega' \Sigma_T^A \omega - r_T^L \tag{10}$$

with Σ_T^A as the covariance matrix, σ_T^A its diagonal, r_T^A the vector of the asset log returns, r_T^L the log liability return, and ω the vector of the asset portfolio weights.[4] All returns are expressed as excess returns over T-bills.

Using Equation (10), we obtain the VAR-implied annualized expected funding ratio return,

$$\mathbb{E}\,(r_T^F) = \frac{1}{T}\left[\omega' \left(\mu_T^A + \frac{1}{2}\sigma_T^A \right) - \frac{1}{2}\omega' \Sigma_T^A \omega - \mu_T^L \right] \tag{11}$$

where $\sigma_T^A \equiv \text{diag}(\Sigma_T^A)$. Further, noting that $r_T^F = (\frac{\omega}{-1})' Hz_T$, we derive from Equation (4) the annualized funding ratio return variance as

$$\mathbb{V}(r_T^F) = \frac{1}{T}[\omega'\Sigma_T^A\omega - 2\omega'\sigma_T^{AL} + \sigma_T^L] \tag{12}$$

Substituting Equations (11) and (12) into Equation (9), we obtain the VAR-implied Lagrangian equation for mean-variance ALM efficient portfolios,

$$L(\omega,\lambda) = \frac{1}{2}\omega'\Sigma_T^A\omega - \omega'\sigma_T^{AL} + \frac{1}{2}\sigma_T^L \\ - \lambda\left(\omega'\left(\mu_T^A + \frac{1}{2}\sigma_T \right) - \frac{1}{2}\omega'\Sigma_T^A\omega - \mu_T^L - \mu_T^F \right) \tag{13}$$

The first-order condition leads to the following description of mean-variance efficient portfolio weights:

$$\omega = \alpha(\Sigma_T^A)^{-1}\left(\mu_T^A + \frac{1}{2}\sigma_T^A \right) + (1-\alpha)(\Sigma_T^A)^{-1}\sigma_T^{AL} \tag{14}$$

with $\alpha = \frac{\lambda}{1+\lambda}$. Given that $\lambda \geq 0$, the result implies that $\alpha \in [0,1]$. We thus obtain a fund separation theorem that allocates a fraction, α, of the wealth to the performance seeking portfolio (PSP) and a fraction, $1 - \alpha$, to the liability-hedging portfolio (LHP).[5] The efficient frontier can then be drawn by letting α vary between zero (which allocates 100% investment to the LHP) and one (which allocates 100% investment to the PSP), and plugging Equation (14) into Equations (11) and (12) in order to derive implied expected returns and implied volatilities, respectively.

We may compare these portfolio allocations to mean-variance efficient portfolios of an asset-only investor who does not take the presence of liability streams into account. As a consequence, the investor focuses on asset returns only, and r_T^F in Equation (10) becomes

$$r_T = \omega'\left(r_T^A + \frac{1}{2}\sigma_T^A \right) - \frac{1}{2}\omega'\Sigma_T^A\omega + r_T^R \tag{15}$$

with r_T^R equal to the return of the risk-free asset (T-bills, in this setting). The expectation in Equation (11) thus becomes

$$\mathbb{E}\,(r_T) = \omega'\left(r_T^A + \frac{1}{2}\sigma_T^A \right) - \frac{1}{2}\omega'\Sigma_T^A\omega + \mu_T^R \tag{16}$$

while the variance in Equation (12) can now be given by

$$\mathbb{V}(r_T) = \omega'\Sigma_T^A\omega + 2\omega'\sigma_T^{AR} + \sigma_T^R \tag{17}$$

where σ_T^{AR} is the covariance vector of the assets with the risk-free asset at horizon T and μ_T^{AR} and σ_T^R the corresponding mean and variance of the risk-free asset. Accordingly, we can solve the Lagrangian for the asset-only problem and obtain the set of mean-variance efficient portfolios $\tilde{\omega}$ as

$$\tilde{\omega} = \alpha(\Sigma_T^A)^{-1}\left(\mu_T^A + \frac{1}{2}\sigma_T^A\right) - (1-\alpha)(\Sigma_T^A)^{-1}\sigma_T^{AR} \quad (18)$$

Again, we obtain a separation result that involves the performance-seeking portfolio as well as a portfolio that captures hedging demand against unexpected changes in interest rate levels (this is the second term in the right side of Equation (18)). Such demand was not present in Equation (14). Because the funding ratio is the ratio of assets over liabilities, the impacts of changes in interest rate levels on the numerator and the denominator cancel out, such that the risk-free rate does not appear in Equation (10). In the empirical section that follows, we will study the behavior of suboptimal strategies (both in the AM and the ALM sense) where the hedging demand in Equation (18) is simply ignored in order to analyze the opportunity costs involved in following a purely myopic portfolio strategy. It should be noted that the portfolios in Equation (18), although efficient in an asset management sense, are not efficient in the ALM space. We will provide, in the following section, ample evidence of the efficiency/ opportunity cost of not taking the presence of liabilities into account in a private wealth management context.

As we have previously argued, our analysis of the life-cycle component of long-term investment decisions provides for asset allocation decisions to be dependent upon the investor's time horizon, a dimension not captured by standard static optimization problems. Although a significant improvement, our approach, which directly follows the seminal work of Campbell, Chan, and Viciera [2003], does not provide the most general form of asset allocation strategies. In fact, three extensions of standard static portfolio allocation models allow for 1) time-horizon dependencies, 2) purely deterministic time dependencies, and 3) time- and state-dependencies. These advances were pioneered by Merton [1969, 1971], who opened a world of opportunities for more subtle dynamic asset allocation decisions involving intertemporal adjustments to the asset mix.

The calculation of optimal intertemporal portfolios is, however, typically very challenging—both analytically and numerically—when the number of state variables exceeds one or two. Therefore, we employ the first extension for simplicity and tractability, focusing on allowing investors with different time horizons to hold different optimal portfolios. The second extension (time dependency without state dependency)—while seeming to offer a good compromise between the conflicting objectives of generality and tractability and often used in the context of so-called target-date funds—cannot be rationalized within a formal asset allocation model (see, for example, Viceira [2007]).

EMPIRICAL ILLUSTRATIONS OF THE BENEFITS OF AN ALM APPROACH IN PRIVATE WEALTH MANAGEMENT

In the empirical applications in this section, we will distinguish between a pension-related objective and a real estate acquisition objective to highlight the importance of properly identifying the appropriate benchmark liability and its impact on optimal portfolios. The nature of the liability stream raises the question of whether a perfect hedge against unexpected shocks in the liability asset is available and how the liabilities should be modeled. In the case of a pension-related objective, it is appropriate to assume that the contractual pension is written in real terms with an actual payment indexed with respect to inflation. In the case of an acquisition of real estate, several real estate–related indices are suitable candidates for the liability benchmark. Thus, to explicitly distinguish between the complete and the incomplete market case, we chose to proxy real estate prices by an investable REIT index.

The illustrations that follow are highly stylized in nature; a number of additional dimensions would have to be addressed in the context of a real-world application of the framework developed in this article. Among these additional dimensions is the need to account for the presence of various tax schedules for different forms of investment. We expect that accounting for differential tax treatment will impact the optimal allocation decisions in a complex manner. It would also be desirable to account for the presence of a variety of constraints (e.g., maximum drawdown limits) and objectives (e.g., bequest motives) that extend beyond the standard expected utility maximization framework used in this article, and which have been found to be relevant to private investors. Finally, one would need to take into account the presence of *flexible* contribution and consumption schedules, versus assuming one or several *predefined* contributions and withdrawals as this analysis does. Designing a general asset allocation model that incorporates realistic tax treatment, a variety of risk budgets, and flexible endogenous contribution and liability schedules is a very desirable objective, but one that cannot be achieved in the context of an analytical model such as the one proposed in this article. Our results are derived under a number of simplifying assumptions, but a number of useful insights can still be gained from our analysis, in particular, that failing to adopt an ALM approach for long-term investment decisions and sticking to the sub-optimal asset-only perspective, will generate very substantial opportunity costs for the private investor.

Pension-Related Objective

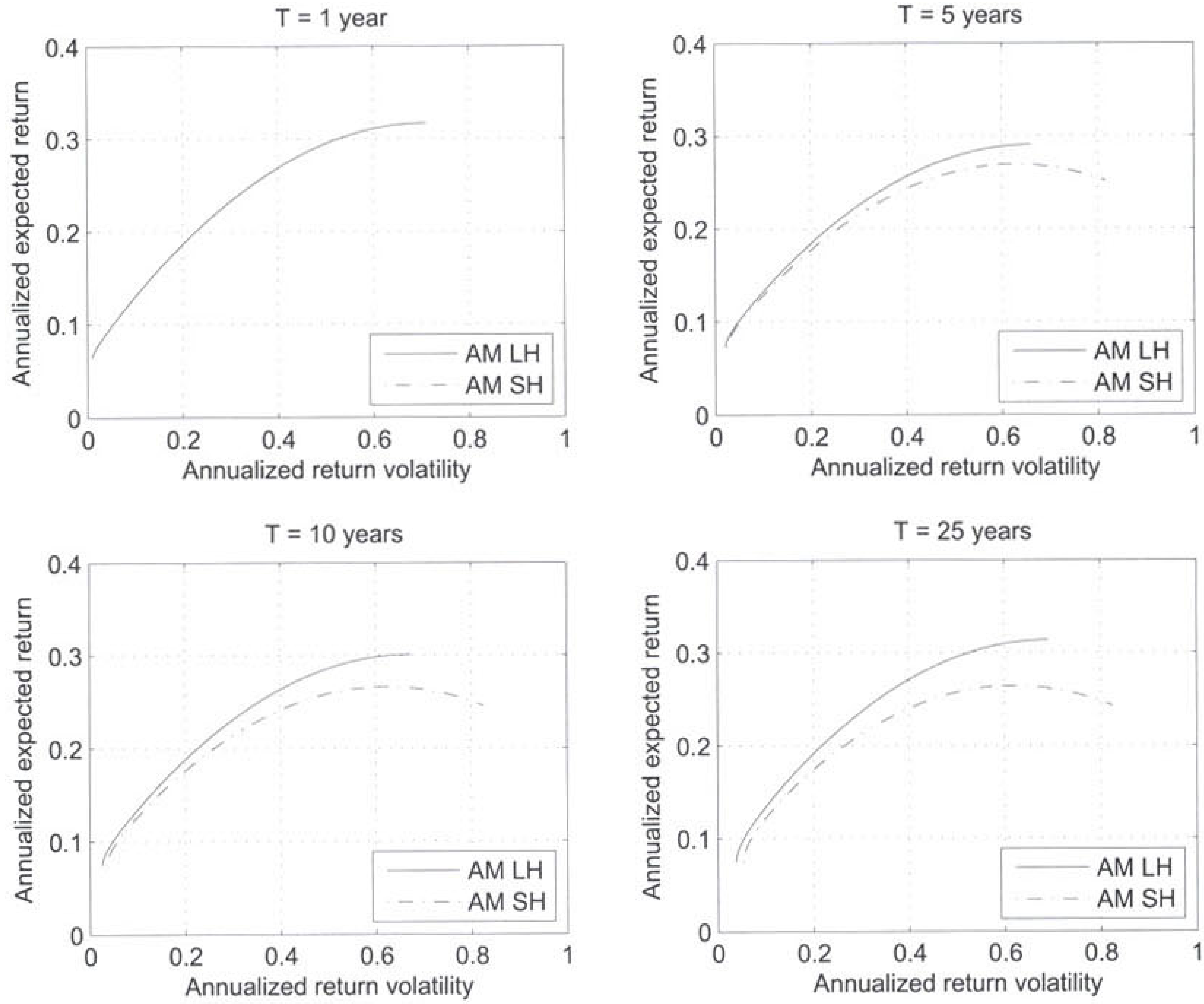
In this subsection, we focus on a pension objective and consider a 65-year-old wealthy individual who is retired.[6] His/her goal is to ensure inflation-protected pension payments, which we normalized at 100 with no loss of generality, at a given horizon date T (we consider T = 1, 5, 10, and 25 years, respectively). To achieve this goal, the individual is prepared to invest a fixed amount of money. We assume that the funding ratio at retirement date is 100%. For each given time horizon, we derive four different efficient frontiers corresponding to 1) the AM objective in Equation (18) where the menu of asset classes does not include the perfect liability-hedging asset and the investor mistakenly uses a short-term one-year horizon when the actual horizon is T years and ignores the hedging demand against interest rate risk (AM SH, or "AM with short horizon"); 2) the AM objective in Equation (18), where the menu of asset classes does not include the perfect liability-hedging asset, but the investor uses the true horizon T when computing the parameters (AM LH, or "AM with long horizon"); 3) the ALM objective in Equation (14), with proper treatment of horizon T, but without the perfect liability-hedging asset in the menu of asset classes (ALM–, or the "incomplete market case"); and 4) the ALM objective in Equation (14) with proper treatment of horizon T and with the perfect liability-hedging asset (an inflation-linked bond) in the menu of asset classes (ALM+, or the "complete market case," because the menu of tradable assets is sufficiently rich to allow for a perfect hedge of liability risk).

The first asset-only approach is consistent with the static approach used in standard asset allocation exercises. The second efficient frontier is an improvement because it allows for time-horizon dependencies, but it still fails to account for the presence of liabilities. The third efficient frontier accounts for both the time horizon and the presence of liabilities, but does not integrate new asset classes with specific liability-hedging properties. The fourth efficient frontier depicts an asset allocation decision that accounts for time-horizon effects and the presence of liabilities and has a specific liability-hedging asset introduced into the asset mix. Separate analysis of the ALM– and ALM+ portfolios allows us to disentangle the two main benefits of the ALM approach, namely, the benefits due to assessing risk and return with respect to the liability benchmark, on the one hand, and the benefits related to the introduction of a liability-hedging instrument, on the other.

We first focus on the benefits obtained by taking into account the investment horizon as opposed to using a standard static model in an asset-only context. To do this, we compute for allocations 1) and 2) the expected value and the variance of the log return on the asset portfolio, following Equations (16) and (17), respectively. We then let α vary over the interval [0, 1] to obtain a representation of these strategies in the AM space, as shown in Exhibit 6. By definition, the AM LH strategy dominates the AM SH strategy in the mean-variance sense, except for the case of a one-year horizon where they are mathematically equivalent. In fact, for T = 5 or 10 years, the performance of the AM strategy with a short horizon is quite similar to that of the AM strategy with a long horizon. Yet, when the horizon is very long (T = 25 years), the opportunity cost of using a static optimization model with a short-term objective is quite substantial.

We then move to an analysis of the additional benefits induced by accounting for the presence of liabilities in addition to the investment horizon. To do this, we obtain expected (log) funding ratios according to Equation (11) and variances for (log) funding ratios from Equation (12) for each set of efficient portfolios corresponding to the situations 2), 3), and 4). Assuming normality of the log returns in Equation (1) and denoting $\mu^F = \mathbb{E}(r_T^F)$ and

EXHIBIT 6
Mean-Variance Efficient Portfolios in the AM Space



Note: These figures plot the efficient frontiers in the ($\mathbb{E}[r_T]$, $\sigma[r_T]$) diagram, where r_T denotes the return on the asset portfolio after T years, as computed in Equations (16) and (17). These frontiers are obtained by letting α vary from zero to one in Equation (18). In the AM LH situation, the investor has a "long-term" horizon equal to T years, while in the AM SH case she applies Equation (18) by mistakenly assuming that T = 1 year.

$\sigma^F = \mathbb{V}(r_T^F)$, we can derive the expected value and the variance of the funding ratio as

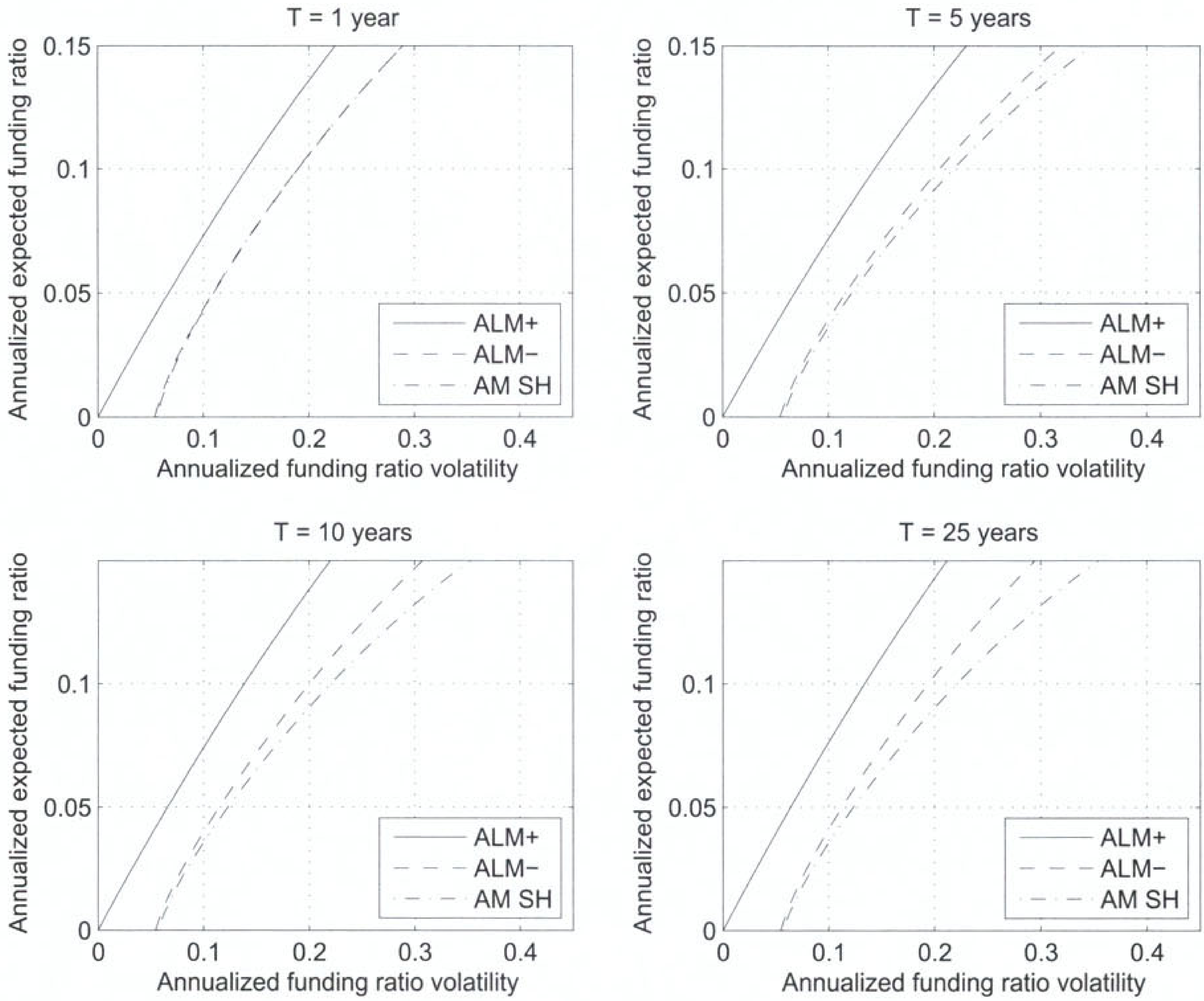
$$\mathbb{E}(FR) = e^{\mu^F + \frac{1}{2}\sigma^F} \tag{19}$$

$$\mathbb{V}(FR) = (e^{\sigma^F} - 1)(e^{2\mu^F + \sigma^F}) \tag{20}$$

We let α vary in Equations (14) and (18) from zero to one and use Equations (19) and (20) to plot the efficient frontiers for various investment horizons. Exhibit 7 shows the three frontiers (ALM with TIPS, ALM without TIPS, and AM SH without TIPS) when the investor goal is based on pension payments related to the Consumer Price Index for four different time horizons (1, 5, 10, and 25 years). The myopic strategy AM SH is strongly dominated by ALM-efficient portfolios, especially when the menu of asset classes available to the ALM investor includes inflation-linked bonds. To get a better sense of the magnitude of the outperformance of ALM-efficient portfolios with respect to asset-only efficient portfolios, Exhibit 8 presents summary statistics for three portfolios on the efficient frontiers in Exhibit 7, corresponding to low, medium, and high target expected returns, respectively. We consider three levels of annualized expected return on the funding ratio, namely, 5%, 10%, and 15%. Shortfall probabilities, $\mathbb{P}(FR < 1)$, are simply derived from

EXHIBIT 7

Mean-Variance Efficient Portfolios in the ALM Space CPI-Indexed Returns



Note: These figures plot the efficient frontiers in the $(\mathbb{E}[r_T^F], \sigma[r_T^F])$ diagram, where $\mathbb{E}[r_T^F]$ and $\sigma[r_T^F]$ denote the annualized expected value and standard deviation of the log funding ratio after T years, respectively as computed in Equations (11) and (12). ALM+ and ALM− are obtained by letting α vary from zero to one in Equation (14). ALM+ refers to the complete market case where TIPS are available for trading, while ALM− refers to the incomplete market setting. AM SH is obtained by mistakenly assuming that $T = 1$ year and letting α vary from zero to one in the myopic portfolio rule.

the relation $\mathbb{P}(FR < 1) = \mathbb{P}(r_T^F < 0)$, which under the assumption of Gaussian returns can be written as

$$\mathbb{P}(r_T^F < 0) = \Phi\left(-\frac{\mu^F}{\sqrt{\sigma^F}}\right) \tag{21}$$

with Φ being the cumulative density function of the standard normal distribution. Expected shortfalls can be derived in analytical form from the Black-Scholes formula as follows.[7]

First, we can decompose the expression for the put price as

$$\begin{aligned}\mathbb{E}\left[e^{-rT}\max(K - S_T, 0)\right] &= e^{-rT}K\mathbb{P}(S_T < K) \\ &\quad - e^{-rT}\mathbb{E}(S_T \mathbf{1}_{\{S_T < K\}}) \\ &= e^{-rT}K\Phi(-d_2) - S_0\Phi(-d_1)\end{aligned} \tag{22}$$

Second, we know that the expected shortfall ES can be written as

$$ES = 1 - \mathbb{E}(S_T \mid S_T < K) = 1 - \frac{\mathbb{E}(S_T \mathbf{1}_{\{S_T < K\}})}{\mathbb{P}(S_T < K)} \tag{23}$$

EXHIBIT 8

Efficient Portfolio Summary Statistics, CPI-Indexed Returns

Horizon (Years)		LOW ALM+	LOW ALM–	LOW AM SH	MEDIUM ALM+	MEDIUM ALM–	MEDIUM AM SH	HIGH ALM+	HIGH ALM–	HIGH AM SH
1	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.71	10.89	10.98	14.12	18.90	18.99	22.46	28.88	28.90
	$\mathbb{P}(FR<1)$	22.77	32.32	32.43	23.91	29.86	29.90	25.22	30.18	30.19
	ES	3.76	6.71	6.77	7.76	10.88	10.93	12.06	15.90	15.92
5	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.83	11.64	12.21	14.42	20.53	21.81	23.01	31.89	34.86
	$\mathbb{P}(FR<1)$	23.18	33.38	34.15	24.36	31.29	32.35	25.72	31.91	33.35
	ES	3.85	7.23	7.62	7.96	11.91	12.72	12.40	17.63	19.29
10	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.57	11.48	12.32	13.85	19.98	22.06	22.00	30.78	35.29
	$\mathbb{P}(FR<1)$	22.41	33.16	34.23	23.51	30.87	32.50	24.78	31.29	33.55
	ES	3.67	7.12	7.69	7.58	11.57	12.87	11.78	17.00	19.52
25	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.43	11.23	12.30	13.43	19.40	22.07	21.16	29.49	35.49
	$\mathbb{P}(FR<1)$	21.72	32.85	34.26	22.76	30.30	32.55	23.95	30.53	33.62
	ES	3.56	6.95	7.69	7.30	11.19	12.88	11.26	16.26	19.63

Note: Expected returns, $\mathbb{E}[r_T^F]$, and return volatilities, $\sigma[r_T^F]$, are annualized percentage values. Shortfall probabilities, $\mathbb{P}(FR < 1)$, and expected shortfalls, ES, are derived from closed-form formulas presented in Equations (21) and (23) and are expressed as percentage values. Nine different setups are studied. ALM+ refers to the ALM objective function under complete market conditions. ALM– denotes the same objective function in incomplete markets, that is, without TIPS, and AM SH is a situation in which the weights follow the myopic portfolio rule, assuming that $T = 1$ year and ignoring the presence of a liability. LOW, MEDIUM, and HIGH refer to exante fixed annualized expected returns on funding ratios of 5%, 10%, and 15% respectively.

Noting that μ^F and σ^F are annualized parameters and setting $S_T \equiv FR, S_0 \equiv 1, K \equiv 1, r \equiv \mu^F + \frac{1}{2}\sigma^F, T \equiv 1$, we can obtain the expected annualized shortfalls from Equation (22) and Equation (23) as

$$ES = 1 - \mathbb{E}(FR \mid FR < 1) = 1 - \frac{e^{\mu^F + \frac{1}{2}\sigma^F}\Phi(-d_1)}{\Phi(-d_2)} \quad (24)$$

with

$$d_1 = \frac{\mu^F + \sigma^F}{\sqrt{\sigma^F}}; \quad d_2 = \frac{\mu^F}{\sqrt{\sigma^F}} \quad (25)$$

Exhibit 8 shows that for a given expected funding ratio, the optimal allocation in the ALM sense dominates optimal allocations in an AM sense in terms of the risk perspective; this holds whether risk is measured in terms of funding ratio volatility, expected shortfall, or probability of shortfall. The result is more pronounced for the more risk-averse investor, which is not surprising given that an AM perspective cannot deliver a sound risk minimization strategy with respect to the liability benchmark. We also obtain that the effect is more pronounced for longer horizons, which is the typical case in a pension-related context. We also confirm that ALM strategies in complete market environments (i.e., when the inflation-linked bonds are included in the asset mix) strongly dominate the corresponding ALM strategies in the absence of a perfect liability-hedging instrument. For example, in the case of a 25-year horizon and a high (15%) expected return target, the volatility of the funding ratio is 21.16% for the ALM strategy in complete markets and 29.49% in an incomplete market, versus 35.49% for the AM strategy.

Exhibit 9 gives the composition of the efficient portfolios for the three strategies, ALM+, ALM–, and AM SH, when the target expected funding ratio is low, high, or medium. We find a substantial amount of leverage in some cases, an effect that is larger when a high expected funding ratio is targeted, implying a more aggressive strategy. When comparing the ALM– and AM SH allocations, we find that the allocation to stocks is smaller in

EXHIBIT 9

Efficient Portfolio Allocations, CPI-Indexed Returns

Horizon (Years)		LOW			MEDIUM			HIGH		
		ALM+	ALM–	AM SH	ALM+	ALM–	AM SH	ALM+	ALM–	AM SH
1	Stocks	0.16	0.07	0.14	0.33	0.22	0.28	0.53	0.40	0.45
	Long Bond	0.09	0.28	0.20	0.20	0.46	0.39	0.32	0.68	0.62
	Commodities	0.30	0.43	0.41	0.63	0.83	0.81	0.99	1.30	1.29
	Real Estate	0.04	0.08	0.07	0.09	0.14	0.13	0.14	0.22	0.21
	TIPS	1.50	0.00	0.00	2.06	0.00	0.00	2.69	0.00	0.00
	T-Bills	–1.09	0.14	0.18	–2.30	–0.65	–0.62	–3.66	–1.59	–1.56
5	Stocks	0.25	0.17	0.14	0.52	0.44	0.28	0.84	0.76	0.47
	Long Bond	–0.08	0.09	0.19	–0.18	0.07	0.39	–0.28	0.05	0.65
	Commodities	0.25	0.38	0.40	0.53	0.74	0.81	0.85	1.18	1.34
	Real Estate	0.11	0.22	0.06	0.23	0.41	0.13	0.36	0.63	0.21
	TIPS	1.63	0.00	0.00	2.32	0.00	0.00	3.11	0.00	0.00
	T-Bills	–1.15	0.13	0.20	–2.43	–0.65	–0.61	–3.88	–1.62	–1.66
10	Stocks	0.32	0.20	0.14	0.67	0.61	0.28	1.07	1.04	0.47
	Long Bond	–0.12	0.01	0.19	–0.25	–0.08	0.39	–0.40	–0.18	0.65
	Commodities	0.23	0.36	0.40	0.49	0.69	0.81	0.79	1.09	1.34
	Real Estate	0.11	0.23	0.06	0.23	0.42	0.13	0.37	0.64	0.21
	TIPS	1.62	0.00	0.00	2.31	0.00	0.00	3.09	0.00	0.00
	T-Bills	–1.16	0.14	0.20	–2.46	–0.64	–0.61	–3.90	–1.60	–1.66
25	Stocks	0.43	0.42	0.14	0.91	0.92	0.28	1.43	1.52	0.47
	Long Bond	–0.11	–0.03	0.19	–0.24	–0.13	0.39	–0.37	–0.25	0.65
	Commodities	0.22	0.32	0.40	0.45	0.62	0.81	0.71	0.98	1.34
	Real Estate	0.11	0.23	0.06	0.23	0.41	0.13	0.37	0.63	0.22
	TIPS	1.61	0.00	0.00	2.27	0.00	0.00	3.00	0.00	0.00
	T-Bills	–1.26	0.06	0.20	–2.63	–0.82	–0.61	–4.14	–1.88	–1.68

Note: This exhibit displays the optimal weights allocated to the various asset classes in different contexts. LOW, MEDIUM, and HIGH refer respectively to a target annualized expected return of 5%, 10%, and 15%. ALM+ refers to the complete market case, where the investor chooses the weights according to formula (14) and has access to TIPS as an asset class only in the hedging part of her portfolio, ALM– refers to the incomplete market case, where she chooses the weights according to the same formula but has no access to TIPS, and AM SH refers to the case where she chooses the weights following the myopic portfolio rule, assumes that $T = 1$ year and cannot trade in TIPS.

ALM– than in AM SH for the short horizon $T = 1$ year. This is in line with the findings presented in the exhibits that suggest stocks have poor hedging properties against inflation in the short run and have substantial volatility (see Exhibit 4). As a result, this makes them relatively undesirable in the liability-hedging portfolio. In contrast, when the horizon lengthens, the inflation-hedging properties of stocks improve and the annualized volatility decreases, making the allocation to stocks higher in ALM– compared to AM SH. This effect is most visible for the longest horizon ($T = 25$ years) and the least risk-averse investor, where the weight allocated to stocks is multiplied by three.

These results are likely to underestimate the inflation-hedging properties of various asset classes, such as stocks, and quite different results would be obtained in the context of a more general model that allowed for the presence of long-term co-integration relationships. Using a vector error correction model (VECM), Amenc, Martellini, and Ziemann [2009] showed that novel forms of investment solutions, including equities, commodities, and real estate, in addition to inflation-linked securities, can be designed to decrease the cost of inflation insurance for long-horizon investors. As a result, it is important to emphasize that investing in inflation-linked instruments is neither the only, nor necessarily the most, cost efficient manner to obtain protection with respect to inflation uncertainty. Indeed the capacity of the inflation-linked securities market is not sufficient to meet the collective demand of institutional and private investors and the OTC inflation-linked derivatives market used by institutional money management is probably not a natural alternative to inflation-linked bonds for private investors. In addition, real returns on inflation-protected securities, negatively impacted by the presence of a significant inflation risk premium, are typically very low which implies that investing in inflation-linked securities, when feasible, is a costly option for pension-hedging motives.

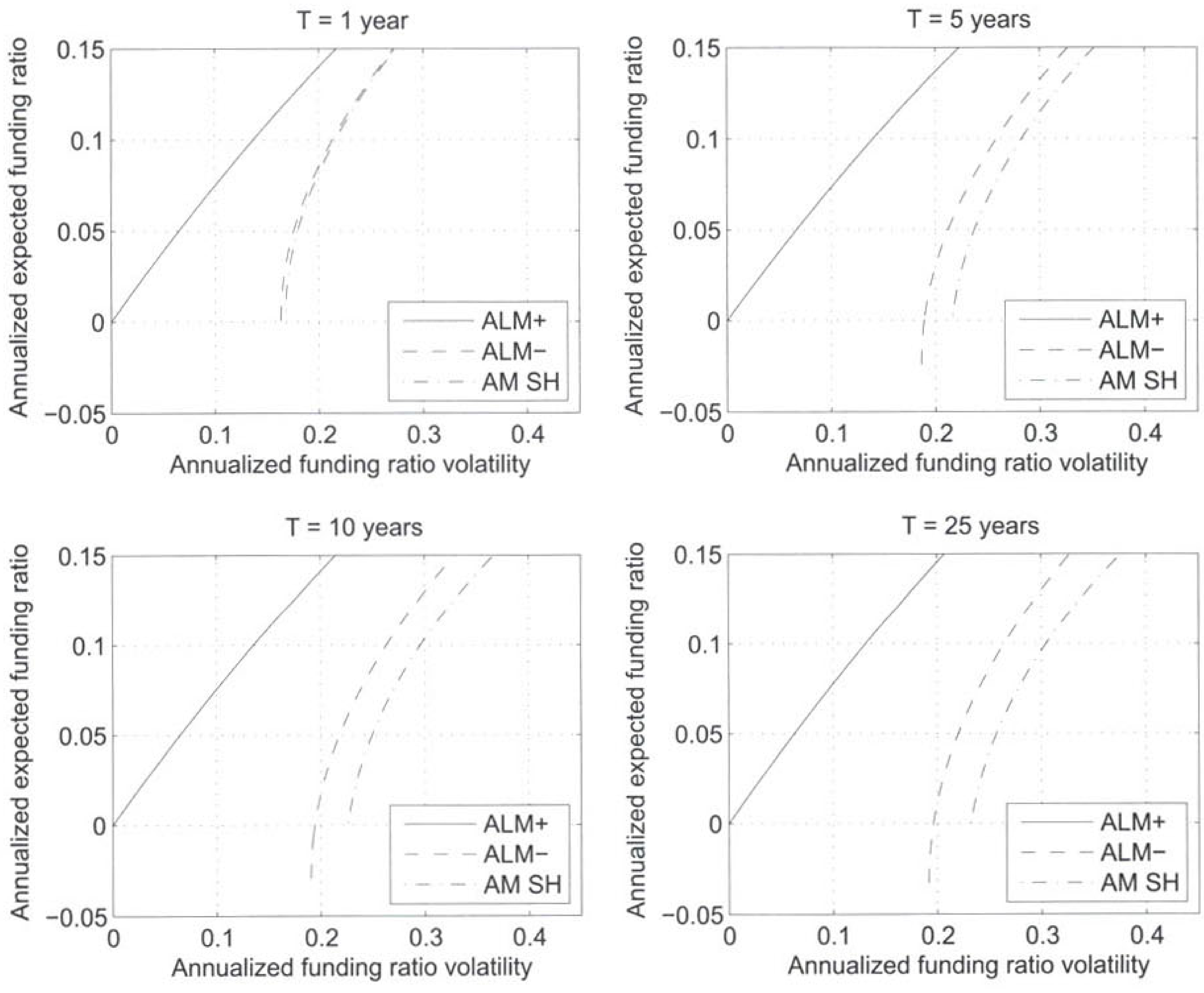
Our results strongly suggest that failing to adopt an ALM approach to long-term investment decisions involves a significant opportunity cost. Our results also suggest that introducing investment vehicles, which enjoy attractive liability-hedging properties, into the menu of asset classes has the potential to add substantial value to investment decision making.

Real Estate Acquisition Objective

We now consider an investor who wishes to invest either a single contribution or a stream of contributions

EXHIBIT 10

Mean-Variance Efficient Portfolios in the ALM Space, RE-Indexed Returns



Note: These figures plot the efficient frontiers in the $(\mathbb{E}[r_T^F], \sigma[r_T^F])$ diagram, where $\mathbb{E}[r_T^F]$ and $\sigma[r_T^F]$ denote the annualized expected value and standard deviation of the log funding ratio after T years, respectively as computed in Equations (11) and (12). ALM+ and ALM– are obtained by letting α vary from zero to one in Equation (14). ALM+ refers to the complete market case where real estate is available for trading, while ALM– refers to the incomplete market setting. AM SH is obtained by mistakenly assuming that $T = 1$ year and letting α vary from zero to one in the myopic portfolio rule.

for a future expenditure (e.g., buy a property in T years), whose current value is normalized at 100, and whose future value evolves according to a stochastic property index value. For simplicity, we assumed that house price increases are perfectly correlated with REIT index returns. Because real estate prices are only imperfectly correlated with a broad-based REIT index, it would be more accurate to introduce a separate series for the dynamics of real estate prices; this, however, would induce a degree of market incompleteness.

Now we will again consider four sets of portfolios: the AM efficient portfolios with a short horizon of one year and a long horizon, $T = 5$, 10, or 25 years, with an asset mix that does not include a real estate–related investment vehicle; the ALM efficient portfolios with a focus on the terminal funding ratio, as opposed to the terminal wealth level, in the absence of the REIT investable index (ALM– efficient frontier); and the ALM efficient portfolios assuming a complete market setting with the REIT index available in the menu of asset classes (ALM+ efficient frontier). To generate comparable portfolios, we consider the improvement in surplus volatility for a given level of expected surplus. We find that the presence of assets allowing investors to span real estate price uncertainty proves to be a key element in improving the efficient frontiers obtained from an ALM perspective. We also confirm that a portfolio efficient in an AM sense is not necessarily efficient in an ALM sense, and vice versa, as was found in the context of the pension-related objective.

Exhibit 10 shows the three efficient frontiers (ALM+, ALM–, and AM SH) when the liability benchmark is given by the real estate index, which reflects the investor's objective of the acquisition of a real estate property. The improvement in efficient frontiers obtained from adopting an ALM perspective is even more spectacular than in the pension-related case. This is presumably due to higher uncertainty in future house price levels compared to future inflation levels. These results are presented in Exhibit 11,

Exhibit 11
Efficient Portfolio Summary Statistics, RE-Indexed Returns

Horizon (Years)		LOW			MEDIUM			HIGH		
		ALM+	ALM–	AM SH	ALM+	ALM–	AM SH	ALM+	ALM–	AM SH
1	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.48	17.56	17.94	13.68	21.31	21.60	21.68	27.23	27.39
	$\mathbb{P}(FR < 1)$	22.14	38.83	39.04	23.23	31.93	32.13	24.46	29.06	29.20
	ES	3.16	11.21	11.45	7.47	12.40	12.58	11.58	14.03	15.03
5	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.66	21.23	23.73	14.03	25.86	28.27	22.34	32.68	35.17
	$\mathbb{P}(FR < 1)$	22.65	40.65	41.67	23.77	34.95	36.15	25.08	32.33	33.51
	ES	3.73	13.55	15.10	7.70	15.19	16.62	11.98	18.08	19.46
10	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.45	21.80	24.95	13.54	26.31	29.61	21.44	32.91	36.71
	$\mathbb{P}(FR < 1)$	21.95	40.92	42.03	23.02	35.21	36.76	24.23	32.41	34.12
	ES	3.58	13.91	15.83	7.38	15.47	17.39	11.43	18.20	20.29
25	$\mathbb{E}[r_T^F]$	5.00	5.00	5.00	10.00	10.00	10.00	15.00	15.00	15.00
	$\sigma[r_T^F]$	6.28	22.04	25.67	13.14	26.43	30.46	20.75	32.78	37.62
	$\mathbb{P}(FR < 1)$	21.32	40.98	42.32	22.33	35.27	37.11	23.49	32.34	34.52
	ES	3.47	14.05	16.28	7.12	15.54	17.88	11.00	18.13	20.79

Note. Expected returns, $\mathbb{E}[r_T^F]$, and returns volatilities, $\sigma[r_F^T]$, are annualized percentage values. Shortfall probabilities, $\mathbb{P}(FR < 1)$, and annualized expected shortfalls, ES, are derived from closed-form formulas presented in Equations (21) and (23). Nine different setups are studied. ALM+ refers to the complete market case in which the investor chooses the weights according to Equation (14) and has access to real estate as an asset class only in the hedging part of her portfolio; ALM– denotes the same objective function in incomplete markets, that is, without real estate; and AM SH relates to a situation in which the weights follow the myopic portfolio rule, assuming that $T = 1$ year and ignoring the presence of a liability. LOW, MEDIUM, and HIGH refer to ex ante fixed annualized expected returns on funding ratios of 5%, 10%, and 15%, respectively.

Exhibit 12
Efficient Portfolio Allocations, RE-Indexed Returns

Horizon (Years)		LOW			MEDIUM			HIGH		
		ALM+	ALM–	AM SH	ALM+	ALM–	AM SH	ALM+	ALM–	AM SH
1	Stocks	0.15	0.11	0.13	0.31	0.28	0.30	0.49	0.47	0.48
	Long Bond	0.09	0.44	0.09	0.19	0.52	0.21	0.30	0.61	0.34
	Commodities	0.28	0.21	0.25	0.59	0.53	0.56	0.94	0.89	0.91
	Real Estate	0.96	0.00	0.00	0.91	0.00	0.00	0.86	0.00	0.00
	TIPS	0.56	0.65	0.50	1.17	1.27	1.14	1.86	1.96	1.84
	T-Bills	–1.03	–0.41	0.02	–2.17	–1.60	–1.21	–3.45	–2.93	–2.58
5	Stocks	0.24	0.33	0.14	0.50	0.61	0.31	0.80	0.93	0.50
	Long Bond	–0.08	0.64	0.10	–0.17	0.56	0.21	–0.27	0.47	0.35
	Commodities	0.24	0.10	0.26	0.51	0.38	0.58	0.81	0.69	0.94
	Real Estate	1.02	0.00	0.00	1.04	0.00	0.00	1.06	0.00	0.00
	TIPS	0.68	1.34	0.53	1.44	2.14	1.17	2.29	3.07	1.90
	T-Bills	–1.10	–1.41	–0.03	–2.31	–2.68	–1.27	–3.68	–4.15	–2.69
10	Stocks	0.31	0.51	0.14	0.64	0.87	0.31	1.02	1.28	0.51
	Long Bond	–0.12	0.60	0.10	–0.24	0.49	0.22	–0.39	0.35	0.36
	Commodities	0.23	0.06	0.27	0.47	0.32	0.59	0.75	0.62	0.96
	Real Estate	1.03	0.00	0.00	1.05	0.00	0.00	1.08	0.00	0.00
	TIPS	0.68	1.42	0.55	1.43	2.23	1.19	2.26	3.16	1.93
	T-Bills	–1.12	–1.59	–0.07	–2.36	–2.91	–1.31	–3.73	–4.41	–2.75
25	Stocks	0.42	0.67	0.15	0.87	1.16	0.32	1.38	1.72	0.52
	Long Bond	–0.11	0.67	0.10	–0.23	0.57	0.22	–0.36	0.45	0.36
	Commodities	0.21	0.04	0.28	0.44	0.27	0.60	0.69	0.54	0.97
	Real Estate	1.03	0.00	0.00	1.06	0.00	0.00	1.10	0.00	0.00
	TIPS	0.66	1.41	0.57	1.38	2.19	1.22	2.18	3.09	1.97
	T-Bills	–1.21	–1.79	–0.10	–2.53	–3.19	–1.36	–4.00	–4.80	–2.82

Note: This exhibit displays the optimal weights allocated to the various asset classes in different contexts. LOW, MEDIUM, and HIGH refer to a target annualized expected return of 5%, 10%, and 15%, respectively, ALM+ refers to the complete market case, where the investor chooses the weights according to Equation (14) and has access to real estate as an asset class only in the hedging part of her portfolio; ALM refers to the incomplete market case, where the investor chooses the weights according to the same formula, but has no access to real estate; and AM SH refers to when the weights are chosen following the myopic portfolio rule, assuming that T = 1 year and, therefore, cannot trade in real estate.

which shows the summary statistics for portfolios on the efficient frontiers plotted in Exhibit 10. We find that for a given expected funding ratio level, the decrease in funding ratio risk (measured in terms of funding ratio volatility, expected shortfall, or shortfall probability) is substantial, especially for more risk-averse investors and longer time horizons. For example, for a 25-year horizon and the more risk-averse investor, the volatility of the funding ratio is almost four times higher with the AM SH objective than with the ALM objective when the REIT index is in the menu of asset classes! As a general comment, we find that the benefits from the introduction of the liability-hedging instrument into the menu of asset classes (going from ALM– to ALM+) generates a marginal benefit that dominates the benefit obtained from replacing the AM objective with an ALM objective without introducing a proper liability-hedging instrument (going from AM to ALM–). Exhibit 12 gives the composition of the optimal portfolios in the ALM+, ALM–, and AM SH cases, and again involves a substantial amount of leverage.

CONCLUSION

In this article, we have shown that a significant fraction of the complexity inherent in optimal asset allocation decisions for private wealth management can conveniently be addressed through the introduction of a single additional state variable, the value of the household liability portfolio, which accounts in a parsimonious and tractable way for investors' specific constraints and objectives. We have also presented a series of numerical illustrations showing how the model we have introduced in this article can be applied in a variety of situations typical to private wealth management problems.

Our analysis has important potential implications for the wealth management industry. It has often been argued that the proximity to investors is the raison d'être of private wealth managers and a key source of competitive advantage. Building on this proximity, private bankers should be ideally placed to account for their clients' liability constraints when engineering an investment solution for them. Most private bankers implicitly promote an ALM approach to wealth management. In particular, they claim to account for the investor's goals and constraints. The technical tools involved, however, are often nonexistent or poorly adapted. Although the private client is routinely asked many questions regarding their current situation, goals, preferences, constraints, and so forth, the resulting service and product offerings mostly boil down to a rather basic classification in terms of risk profiles. In this article, we have provided a framework that suggests asset-liability management is an essential improvement in private wealth management, which allows private bankers to provide their clients with investment solutions and asset allocation advice that truly meet their needs.

Broadly speaking, our analysis has shown that adopting an ALM perspective to private wealth management generates two main benefits. First, adopting an ALM approach has a direct impact on the selection of asset classes. In particular, it leads to a focus on liability-hedging properties of various asset classes, a focus that is by definition absent from an asset-only perspective. Second, adopting an ALM approach leads to defining risk and return in relative, versus absolute, terms, with the liability portfolio used as a benchmark, or numeraire. This is a critical improvement over asset-only asset allocation models, which fail to recognize that changes to asset values have to be analyzed in comparison to changes in liability values. In other words, private investors do not have utility over terminal wealth per se, but over the purchasing power of terminal wealth expressed in terms of various investment and/or consumption goals and objectives, such as preparing for retirement or preparing for a real estate acquisition.

Our research can be extended in a number of directions. First, it would be desirable to incorporate the impact of taxes in the analysis of optimal asset allocation decisions. Tax optimization is arguably one of the key sources of added value in private wealth management. It is, however, typically very challenging to account for the detailed features of tax regulations, which vary significantly across countries, in the context of a formal optimal allocation model, especially given that tax treatment depends on trading behavior and portfolio turnover.[8] Other elements left for further research are the introduction of flexible contribution/withdrawal decisions and the extension of the asset allocation model to more general forms of state-dependent optimal allocation strategies. Finally, and perhaps most importantly, it could be valuable to apply the ALM approach to private wealth management when the investor has a behavioral objective. One challenge is that recent advances in behavioral finance, although they provide very useful insights into investor behavior, do not provide much guidance toward the design of a formal normative analysis of optimal asset allocation decisions. A possible approach would be to capture some of this complexity by introducing a set of suitably specified investor-dependent goals and constraints within the standard expected utility maximization paradigm. These questions, as well as other possible extensions, are also left for further research.

ENDNOTES

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[1]A notable exception is Wilcox and Fabozzi [2009] who argue in favor of integrating the present value of future saving and spending plans in the assessment of an investor's wealth.

[2]The use of a constant-maturity approach in modeling TIPS returns is consistent with various investment horizons if the real yield curve is flat.

[3]An alternative approach to ordinary least squares (OLS) for estimating VAR models is the Yule-Walker (YW) methodology. This methodology consists of using "presample" observations of the state vector z to estimate the partial autocovariances; then, the system of the Yule-Walker equations, relating theoretical autocovariances to VAR parameters, is inverted, which generates estimates for these parameters (see Lütkepohl [1993] for more details, as well as Steehouwer [2005] for a modified form

of the YW estimator in which the *biased* versions of the estimated autocovariances are used). The main reason we prefer using the OLS estimator is that it is unclear how restrictions on the model parameters can be implemented when inverting Yule-Walker equations. Such restrictions are easily taken into account, however, when the VAR is estimated equation by equation. In the case of a VAR(1), using the YW approach involves the use of only one additional pre-sample observation. As a result, even if we were to estimate an unrestricted form of the VAR model, differences in the two competing estimation methods would be expected to be small for larger sample sizes.

[4]Note that we omit the time-index t because we assume a stationary system.

[5]With our choice of notation, the weights allocated to risky assets in the PSP and LHP portfolios have not been normalized and do not add up to 100%; the remainder is invested in cash. Hence, we obtain a three-fund separation theorem involving the PSP and the LHP and the risk-free asset.

[6]Similar results could be obtained for a younger investor who was preparing for retirement by making contributions to her pension portfolio until her retirement date.

[7]One key difference is that probabilities are taken under the risk-adjusted measure in an option pricing context and are taken under the historical measure in a risk parameter estimation context.

[8]Moreover, most industrialized and developing countries have tax incentives to encourage retirement savings. Examples of countries that offer tax-advantaged savings accounts include Australia, Canada, Germany, Italy, The Netherlands, and the U.K., and most of these countries permit tax-deductible contributions.

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